

A finite dimensional algebra of the diagram of a knot

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Abstract

To a regular projection of a knot we associate a finite dimensional non-commutative associative algebra which is self-injective and special biserial.

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1 Introduction

Let D be an oriented knot diagram, that is a regular projection of a knot to the plane where each crossing is given with the information of which part of the knot undercrosses the other one; this appears in D by interrupting the part of the diagram which is under-crossing.

In this note we associate to D a finite dimensional algebra over a field k , presented by a quiver with relations deduced from the knot. In other words we associate to D a k -category given by a presentation; it has a finite number of objects corresponding to the crosses, and finite dimensional vector spaces of morphisms. The algebra of the diagram is the direct sum of all the morphisms of the k -category with product induced by the composition of the category or, equivalently, is the path algebra of the quiver modulo the two-sided ideal generated by the relations.

This algebra of the knot projection is Morita reduced, special biserial and self-injective. As such it is not invariant under Reidemeister moves since its dimension changes.

The main purpose of this note is the description of this family of algebras. They can be of interest in order to test homological conjectures or to analyze representation theory aspects. On the other hand one may expect information on the knot via the algebra of a diagram, as well as links between knot theory and the representation theory of finite dimensional algebras.

On some examples this algebra happens to admit a connected grading by the fundamental group of the knot.

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2 Quiver and relations of an oriented knot diagram

A quiver is an oriented graph Q given by two finite sets, Q_0 (vertices) and Q_1 (arrows) and two maps $s, t : Q_1 \rightarrow Q_0$ assigning to each arrow a the source and target vertices $s(a)$ and $t(a)$. Consider the vector space kQ with basis the set of oriented paths of Q including the trivial ones given by the vertices. The quiver path algebra is kQ equipped with by the product on paths induced by their concatenation if it can be performed and 0 otherwise. This way the vertices provide a complete set of primitive orthogonal idempotents. Note that Q_0 needs to be non empty in order to get an algebra.

Consider F the two sided ideal generated by the arrows. Clearly $kQ/F \simeq kQ_0$ where kQ_0 is the commutative semi-simple algebra of the set Q_0 . In case Q has no oriented cycles F is the Jacobson radical of kQ . A well known theorem of P. Gabriel states that for each finite dimensional Morita reduced algebra Λ over an algebraically closed field there is a unique quiver Q_Λ such that Λ is isomorphic to a quotient kQ_Λ/I where I is an admissible two-sided ideal, that is I contains F^2 and is contained in F^n for some positive integer n . Note that I is not unique in general, while Q_Λ is the Ext quiver with vertices the iso-classes of simple modules and as many arrows between two simples than the dimension of Ext_Λ^1 between them.

Let D be an oriented knot diagram. An **arc** of D is obtained by following the diagram according to its orientation from an under-crossing to the next one, in other words an arc is a connected component of D . A **segment** of the diagram is obtained by following the diagram according to its orientation, from one crossing to a crossing with no crossings in-between.

Definition 2.1 The quiver Q_D has set of vertices the crossings of D . Note that this requires to have at least one crossing in D in order that $(Q_D)_0 \neq \emptyset$. Each segment provides an arrow having source and target the corresponding crossings. We say that the source of a an arrow is negative or positive according if the segment starts respectively by under-crossing or by over-crossing. Similarly the target vertex of an arrow is also negative or positive.

Remark 2.2 There are two specific oriented cycles at each vertex e as follows:

1. α_e starts at e by the arrow with positive source e and successive arrows by browsing the oriented diagram until reaching the arrow having target e with negative sign,
2. Similarly, β_e starts at e but by the negative source arrow and ends with the arrow with positive target e .

Definition 2.3 The two fundamental cycles at e are as follows: the over-crossing (or positive) one $\gamma_e^+ = \beta_e \alpha_e$ and the under-crossing (or negative) one $\gamma_e^- = \alpha_e \beta_e$.

Lemma 2.4 The length of the fundamental cycles do not depend on the vertex and equals the number of arrows n_D of Q_D .

Let $\tau : Q_0 \rightarrow k^\bullet$ be a map. For instance $\tau(e) = q^{l(e)}$ where $q \in k^\bullet$ and $l : Q_0 \rightarrow \mathbb{Z}$ is some map given for example by the length of α_e . We consider a two-sided ideal I_τ of kQ_D generated by two kind of relations as follows.

- Type I : Let ba be a path of length 2 in Q_D and let $e = t(a) = s(b)$ be its middle vertex. In case the signs at e of a and b are different (that is if ba is not a follow-up of two segments of the diagram) then ba is a generator. This way each vertex of the quiver provides two generators.
- Type II : The elements $\alpha_e \beta_e - \tau(e) \beta_e \alpha_e$ for all the vertices e .

Definition 2.5 *The algebra of the diagram D with respect to τ is $\Lambda_{D,\tau} = kQ_D/I_\tau$.*

Proposition 2.6 *The algebra $\Lambda_{D,\tau}$ is finite dimensional. A basis is given by the positive fundamental cycles and the non-zero paths of length strictly less than n_D , that is the paths made by following-up the segments and having strictly less than n_D segments.*

Proof. The proof relies on the observation that a path δ of length $n_D + 1$ is zero in the algebra. Indeed let $\delta = a\gamma$ where γ is a fundamental cycle. If the source sign of a is not the same than the target sign of γ then $\delta \in I_\tau$. Otherwise let γ' be the other fundamental cycle, then $a\gamma$ and $a\gamma'$ are equal up to a non-zero element of k and the latter is in the ideal.

Remark 2.7 *The number of vertices of Q change through the first and second Reidemeister moves.*

Let J_D be the two-sided ideal generated by the relations of type I and type II', where type II' is the set of all paths of length $n_D + 1$. Instead of $\Lambda_{D,\tau}$ a monomial algebra $\Xi_D = kQ_D/J_D$ can be considered which we call the **monomial algebra of the diagram**. A larger basis than before is provided by all the non-zero paths of length strictly less than $n_D + 1$.

Example 2.8 *Let D be the diagram of the trivial knot with one crossing. Then the algebra $\Lambda_{D,\tau}$ is $k\{a, b\} / \langle ab - qba \rangle$ for $q \in k^\bullet$. This algebra provides the first example for a negative answer to Happel's question, see [3]. More precisely its global dimension is infinite but for q not a root of unity it has zero Hochschild cohomology in degrees large enough (in fact starting at degree 3). Nevertheless this algebra verifies Han's conjecture, see [16], namely its Hochschild homology is non zero in arbitrarily large degrees (in fact in all degrees).*

3 Properties

We recall first the definition of a family of algebras which arose in representation theory of finite dimensional algebras.

Definition 3.1 Let Q be a quiver and I a two sided admissible ideal generated by a set of relation ρ . Then (Q, ρ) is called **special biserial** (see for instance [13]) if it verifies the following conditions:

1. Any vertex of Q is the source of at most two arrows and is the target of at most two arrows.
2. If two different arrows c and d start at the target of an arrow a then at least one of the paths ca or da is in ρ .
3. If two different arrows a and b end at the source of an arrow c then at least one of the two paths ca or cb is in ρ .

As mentioned by C.M. Ringel in [13] special biserial algebras were first considered by I.M. Gelfand and V.A. Ponomarev in [15]. Blocks of a group algebra with cyclic or dihedral defect group are special biserial. As a consequence of [15] special biserial algebras are of tame representation type, in other words their indecomposable modules can be classified, see also [23, 12]. Precise conditions are given in [1] for the vanishing of the first Hochschild cohomology of a special biserial algebras (which in turn implies that the cohomology in degrees larger than 1 also vanishes).

The following result is immediate:

Proposition 3.2 *The algebra (or the monomial algebra) of the diagram of a knot is special biserial.*

We recall that an algebra is **self-injective** if it admits a non degenerate bilinear form $\beta : \Lambda \times \Lambda \rightarrow k$ which is associative, that is $\beta(xy, z) = \beta(x, yz)$ for any triple of elements (x, y, z) in Λ . Associated to β there is a linear map $t : \Lambda \rightarrow k$ given by $t(x) = \beta(x, 1)$ which is a free generator of the left Λ -module $\text{Hom}_k(\Lambda, k)$.

Theorem 3.3 *Algebras of diagrams of knots are self-injective.*

Proof. Let δ be a positive length basis path of $\Lambda_{D, \tau}$ (according to Lemma 2.6) with source vertex e . We define δ' to be the path such that $\delta'\delta = \gamma_e^\epsilon$, where γ_e^ϵ is the fundamental cycle and ϵ is the sign of e at the first arrow of δ . Note that $(\gamma_e^+)' = e$. Moreover for each vertex e we put $e' = \gamma_e^+$.

In case δ_1 and δ_2 are basis paths such that $\delta_2 \neq \delta_1'$ we put $\beta(\delta_2, \delta_1) = 0$.

If $\delta_2 = \delta_1'$ we consider two cases :

1. In case δ_1 is a vertex or if the source of the first arrow of δ_1 is positive, then $\beta(\delta_1', \delta_1) = 1$.
2. If the sign of the source e of the first arrow of δ_1 is negative, then $\beta(\delta_1', \delta_1) = \tau(e)$.

The only difficulty for verifying that β is associative arises when δ_1 has a first arrow with negative source e . We need to prove that

$$\beta(\delta_1', \delta_1) = \beta(\delta_1' \delta_1, e).$$

We have defined $\beta(\delta'_1, \delta_1) = \tau(e)$ while

$$\beta(\delta'_1, \delta_1, e) = \beta(\gamma_e^-, e) = \beta(\tau(e)\gamma_e^+, e) = \tau(e)\beta(\gamma_e^+, e) = \tau(e)$$

There is no difficulty for showing that β is non-degenerated.

Remark 3.4 *The class of special biserial self-injective algebras has been studied by K. Erdmann and A. Skowroński with respect to Euclidian components of the stable Auslander-Reiten quiver, see [14]. See also the work by Z. Pogorzały [21] concerning stable equivalence of this class of algebras. Precise computations of the Hochschild cohomology of certain self-injective special biserial algebras are performed in [22].*

4 Gradings

The fundamental group *à la Grothendieck* of a k -category has been considered in [7, 8, 9, 10, 11]. Previously a fundamental group depending on a presentation by a quiver with relations has been studied in relation with representation theory, see for instance [2, 20, 4, 5, 17, 18, 19]. The main tool for the theory of the intrinsic fundamental group are the connected gradings as follows.

Definition 4.1 *Let \mathcal{B} be a small k -category. A **grading** X of \mathcal{B} with structural group $\Gamma(X)$ is firstly a direct sum decomposition of each vector space of morphisms indexed by elements of $\Gamma(X)$ – a direct summand with index g of this decomposition is called a homogeneous component of degree g , and a non zero morphism in this component is said to be homogeneous of degree g . Secondly the composition of two homogeneous morphisms is homogeneous with degree the product of the degrees.*

The precise definition of homogeneous walks is given for instance in [11]. Roughly each homogeneous morphism φ of degree g provides a virtual one $(\varphi, -1)$ with reversed source and target vertices and of settled degree g^{-1} . A **homogeneous walk** is a sequence of concatenated (virtual or not) homogeneous morphisms; its degree is the product of the degrees.

The grading X is **connected** if between two fixed objects and for any element of $g \in \Gamma(X)$ there exist a homogenous walk relying the objects having degree g . In this case the smash product provides a connected category which is a Galois covering of \mathcal{B} , see [6].

The fundamental group is obtained by considering all the connected gradings of \mathcal{B} and coherent families of elements of the structure groups with respect to morphisms of gradings, see [11].

We recall the following convention: a crossing of D is positive if following the diagram according to the orientation the under line of the diagram goes from right to left. It is negative otherwise.

Let D be the diagram of an oriented knot. Recall that the fundamental group of the complement of the knot with base point at the infinity is generated by the loops which passes just under each portion of the knot corresponding to an arc of the projection.

Definition 4.2 Let Q_D be the diagram of a knot. The grading of Q_D by the fundamental group of the knot is as follows: given an arrow, consider the corresponding segment and the arc to which it belongs. In case the crossings at the vertices of the arrow have the same sign, the degree of the arrow is the generator of the fundamental group corresponding to the arc. Otherwise the degree is trivial.

The verification of the following result is not difficult using the Wirtinger presentation of the fundamental group of the knot.

Example 4.3 For the usual diagrams of the trefoils knots or of the figure-height knot the two-sided ideal I_+ is homogeneous and the resulting grading for the algebra of the diagram is connected.

Nevertheless it seems that for the diagram of the knot 6_2 (see the for instance the Knot Atlas) the ideal is not homogeneous.

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